Lecture 6
Space charge impedance. Smooth transitions. Diffraction impedance.

January 24, 2019

Lecture outline

- Model wakes: resistive, inductive, and capacitive
- Indirect space charge force
- Longitudinal space charge
- Yokoya's formulas for a smooth collimator
- Diffraction model for the impedance

Resistive wake

There are several general models of the *longitudinal* wake. Sometimes it is convenient to use model wakes that approximate some part of reality. These wake functions are singular, and the Kramers-Kronig relations may not hold for them. For each wake, we plot the bunch wake (4.2), $W_\ell(s)$, for a Gaussian bunch distribution (shown by the dashed line).

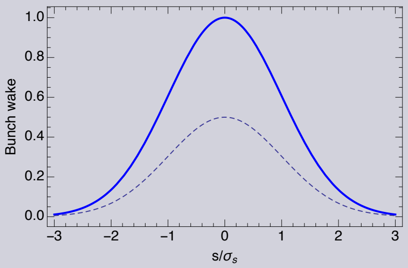
1. The *resistive wake*, R is the resistivity

$$w_\ell(s) = Rc\delta(s) \tag{6.1}$$

$$Z_\ell = R$$

$$W_\ell(s) = Rc\lambda(s)$$

The resistive bunch wake has the same shape as the bunch distribution.



Do not confuse this wake with the *resistive wall* wake. Note[20].

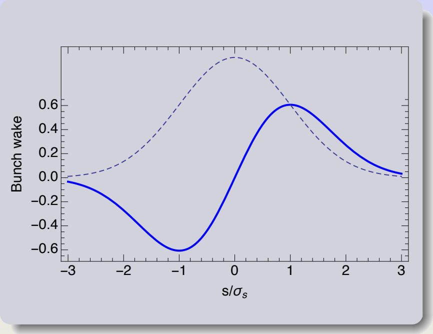
[20] In order to not violate the causality, the delta function is assumed to be slightly shifted to positive s, i.e., $\delta(s-\epsilon)$ with $\epsilon \to 0$.

Inductive wake

2. The *inductive wake*, L is the inductance,

$$w_\ell(s) = Lc^2\delta'(s) \qquad (6.2)$$
$$Z_\ell = -i\omega L$$
$$W_\ell(s) = -Lc^2\lambda'(s)$$

Typically $L > 0$. The bunch head at positive s. No average energy loss.



This type of wake is often a good approximation for long bunches propagating in a vacuum chamber with large conductivity, when the beam energy losses can be neglected (see below). The inductive impedance often approximates the limit of small frequencies ω.

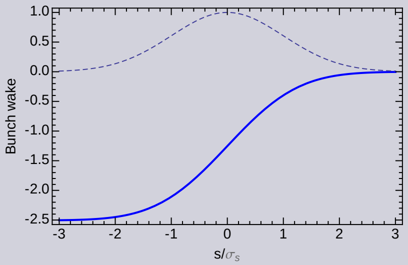
Capacitive wake

3. The *capacitive wake*: $h(s)$ is the step function, C is the capacitance,

$$w_\ell(s) = Ch(s) \tag{6.3}$$

$$Z_\ell = -\frac{C}{i\omega}$$

$$W(s) = C\int_s^\infty ds'\lambda(s')$$



The bunch head is at positive s.

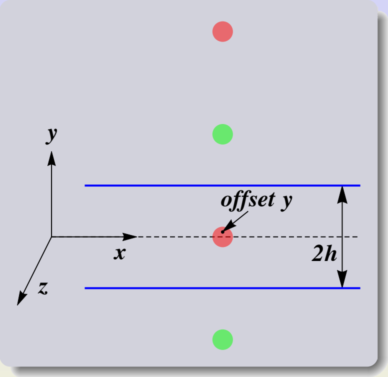
Transverse space charge effects

We calculated the transverse fields and the transverse force inside a relativistic beam with uniform charge in L2, Eq. (2.13). We assumed the beam in free space, however, the same result is valid if the beam is inside a round pipe (we only used the symmetry of the problem in the derivation). This force is called the *direct space charge force*. It scales as γ^{-2} and becomes small for relativistic beams.

The situation is different if the vacuum chamber is not round. It turns out that the *averaged over time* beam current, $\langle I \rangle$, has a contribution to the transverse force that does not depend on γ. Typically $\langle I \rangle \ll I_{\mathrm{peak}}$, but in the limit $\gamma \gg$ this field can be important. It is often called the *indirect space charge*.

We consider here an approximation of two parallel plates. We will also assume that the beam current does not depend on time, and hence the magnetic field penetrates through the wall.

Transverse space charge effects



Consider a beam of radius a with λ is the number of particles per unit length (λ is assumed constant). The beam is located at the center line. Consider a particle that has offset y. In addition to the electric field (2.12) there is electric field from the image charges acting on the particle, see the figure.

$$E_y = \frac{e\lambda}{2\pi\epsilon_0}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y}-\frac{1}{2nh-y}\right]$$

$$= -\frac{e\lambda}{\pi\epsilon_0}y\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2nh)^2-y^2} \tag{6.4}$$

Effect of conducting and magnetic screens

We are interested in the limit $y \ll h$, and can neglect y^2 under the sum. The result is $\left(\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2} = -\pi^2/12\right)$

$$E_y = \frac{\pi}{48}\frac{e\lambda}{\epsilon_0 h^2} y \tag{6.5}$$

From the divergence equation $\partial E_x/\partial x + \partial E_y/\partial y = 0$ one can find electric field E_x near the axis:

$$E_x = -\frac{\pi}{48}\frac{e\lambda}{\epsilon_0 h^2} x \tag{6.6}$$

Multiplied by the charge, these terms add to the self-force Eq. (2.13). Note that in contranst to Eq. (2.13) there is no γ^{-2} suppression in the indirect force.

There is no contribution from magnetic images, unless there are high μ material outside of the pipe.

Longitudinal space charge effects

We now calculate the longitudinal field E_z of a beam in a round, perfectly conducting pipe (in L2 we estimated the longitudinal field in free space, Eq. (2.15)). Consider a beam of charge Q that has the density distribution $n(r, z)$ in the form

$$n(r, z) = \nu(r)\lambda(z)$$

We assume that $\nu(r)$ is normalized by $2\pi \int_0^b r\, \nu(r)\, dr = 1$, where b is the pipe radius; then λ is the number of particles per unit length.

We go into *the beam frame*. The beam density there is $(r' = r)$

$$n'(r, z') = \nu'(r)\lambda'(z') \tag{6.7}$$

We then solve the equation for the potential $\phi'(r, z')$ in the beam frame

$$\Delta\phi' = -\frac{Q}{\epsilon_0} n'$$

We assume a *long wavelength* perturbation, much longer than the pipe radius b, and neglect the second derivative $\partial^2\phi'/\partial z'^2$ in the Laplacian,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi'}{\partial r} = \frac{Q}{\epsilon_0}\nu'(r)\lambda'(z') \tag{6.8}$$

It is clear that $\phi' \propto \lambda'(z')$. We solve the radial part of the equation using the method of Green functions.

Longitudinal space charge effects

The Green function for the last equation is the solution of

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial G(r,\xi)}{\partial r} = \delta(r-\xi), \tag{6.9}$$

with the boundary condition $G(b,\xi) = 0$ and $G(0,\xi)$ finite. The solution is

$$G(r,\xi) = \begin{cases} \xi \ln \frac{\xi}{b} & \text{if } r < \xi \\ \xi \ln \frac{r}{b} & \text{if } r > \xi \end{cases} \tag{6.10}$$

With this Green's function we find the field

$$\phi'(r,z') = -\frac{Q}{\epsilon_0}\lambda'(z')\left(\int_0^r \xi\, d\xi\, \nu'(\xi) \ln\frac{r}{b} + \int_r^b \xi\, d\xi\, \nu'(\xi) \ln\frac{\xi}{b}\right) \tag{6.11}$$

Space charge in a perfectly conducting round pipe

The electric field in the beam frame is

$$E'_z = -\frac{\partial \phi'}{\partial z'} \tag{6.12}$$

We now need to transform this into the lab frame. We have $E_z = E'_z$, $\nu(r) = \nu'(r)$ and $z' = \gamma z$, so that $\lambda'(z') = \lambda(z'/\gamma)/\gamma$

$$\frac{\partial \lambda'}{\partial z'} = \frac{1}{\gamma}\frac{\partial \lambda}{\partial z'} = \frac{1}{\gamma^2}\frac{\partial \lambda}{\partial z} \tag{6.13}$$

so

$$E_z(r,z) = \frac{Q}{\epsilon_0 \gamma^2}\frac{d\lambda(z)}{dz}\left(\int_0^r \xi\, d\xi\, \nu(\xi) \ln\frac{r}{b} + \int_r^b \xi\, d\xi\, \nu(\xi) \ln\frac{\xi}{b}\right) \tag{6.14}$$

For a beam of radius a with uniform radial distribution in the region $r < a$, $\nu = 1/\pi a^2$, the expression in the bracket is

$$-\frac{1}{2\pi}\left(\ln\frac{b}{a} + \frac{1}{2} - \frac{r^2}{2a^2}\right) \equiv -\frac{1}{2\pi}\Lambda \tag{6.15}$$

Space charge in a perfectly conducting round pipe

We have

$$E_z(r, z) = -\frac{Q}{2\pi\epsilon_0\gamma^2}\Lambda\frac{d\lambda(z)}{dz} \tag{6.16}$$

Strictly speaking, E_z depends on r, but this dependence is weak, and often is neglected setting $r = 0$, then

$$\Lambda = \ln(b/a) + 1/2$$

Comparing with Eqs. (6.2) we see[21] that the wake function (per unit length) is the derivative of the delta function

$$w_\ell(s) = -\frac{1}{2\pi\epsilon_0\gamma^2}\Lambda\frac{d\delta(s)}{ds}$$

This is not our standard wake—it depends on the transverse beam size and its energy.

Instead of setting $r = 0$, it makes more sense though to integrate over the cross section of the beam.

[21] We use the relation $W_\ell = -E_z/Q$ with W_ℓ the bunch wake per unit length.

Space charge in a perfectly conducting round pipe

The space charge impedance (per unit length) is

$$Z_\ell = i\omega\frac{1}{2\pi\epsilon_0\gamma^2 c^2}\Lambda = iZ_0\Lambda\frac{\omega}{2\pi c\gamma^2}$$

which is the inductive impedance with *negative* inductance.
Recall that we assumed long bunch length in the beam frame. This means

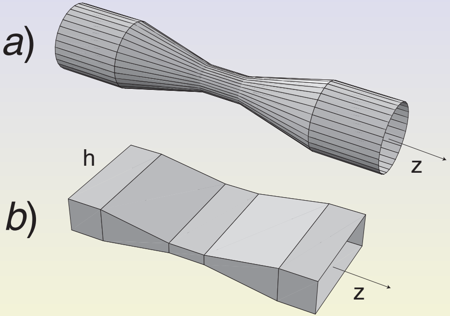
$$\sigma_z \gg \frac{b}{\gamma}$$

For the impedance, this condition translates into the requirement

$$\frac{\omega}{c} \ll \frac{\gamma}{b}$$

Yokoya's formulas for wake of a smooth transition

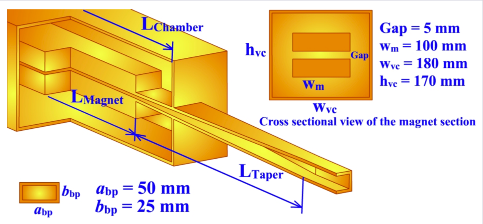
To remove halo particles from the beam one often uses collimators. To lower the collimator impedance one can try to taper the collimator jaws to get a gradual transition from a large to a small aperture and back.



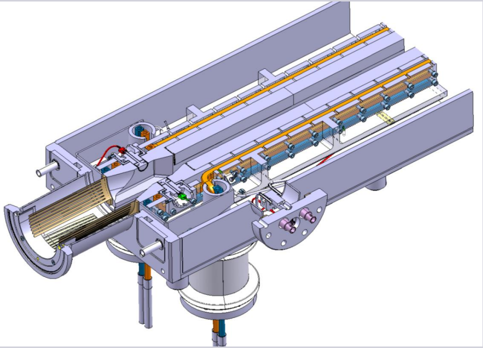
Another example is transitions to and from undulators (that have small gaps) in light sources.

Practical examples of tapers and collimators

Taper to an undulator (from NSLS-II CDR).



An LHC collimator.



Yokoya's formulas for wake of a smooth transition

The impedance of a smooth *round* tapered transition was calculated by K. Yokoya[22] *in the limit of low frequencies*.

$$Z_\ell(\omega) = -\frac{i\omega Z_0}{4\pi c} \int_{-\infty}^{\infty} dz (a')^2 \tag{6.17}$$

and the transverse impedance

$$Z_t(\omega) = -\frac{iZ_0}{2\pi} \int_{-\infty}^{\infty} dz \left(\frac{a'}{a}\right)^2 \tag{6.18}$$

where $a(z)$ is the pipe radius as a function of z, and the prime denotes the derivative with respect to z, $a' = da/dz$. These formulas assume that the taper angle is small, $\alpha \equiv |a'(z)| \ll 1$.

[22] K. Yokoya, "Impedance of Slowly Tapered Structures," Preprint CERN SL/90-88 (AP) (1990)

Yokoya's formulas for wake of a smooth transition

Another condition of applicability of Yokoya's formula is

$$\alpha k a \ll A,$$

$k = \omega/c$, and A is a numerical factor of order of unity. For a bunch of length σ_z, the characteristic value of k in the beam spectrum is equal to σ_z^{-1}. Applying Eq. (6.17) to a conical transition with the conical angle α that connects two pipes of radii a_1 and a_2 ($a_2 > a_1$) gives the following result:

$$Z_t = -\frac{iZ_0}{\pi a_{\mathrm{av}}} \frac{\epsilon \tan \alpha}{1 - \epsilon^2} = -\frac{iZ_0}{2\pi} \frac{(a_2 - a_1)^2}{a_1 a_2} L$$

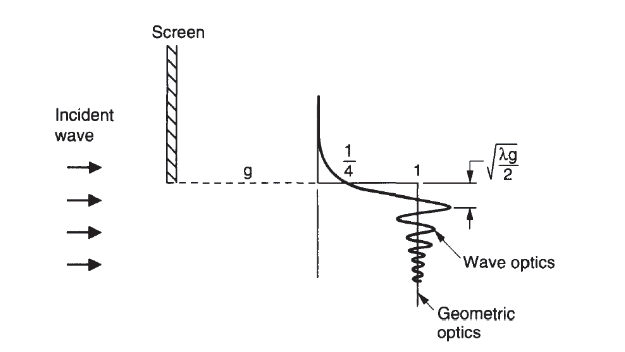
where $a_{\mathrm{av}} = (a_1 + a_2)/2$ and $\epsilon = (a_2 - a_1)/(a_2 + a_1)$ and L is the length of the transition. More results on this subject can be found in the review paper[23].

Small-angle collimators are not easy to simulate in computer programs, because some codes represent a smooth pipe wall by a sequence of small steps in r.

[23] G. Stupakov, PAC09, page 4270, Vancouver, Canada, 2009.

Fresnel diffraction (from A. Chao's book)

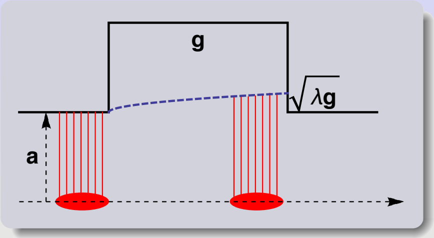
Using the connection between the energy loss and impedance we can estimate impedance of a so called *pillbox cavity* at high frequencies.



We want to apply the idea of diffraction to the field of an ultrarelativistic beam — the*diffraction model*. It works in the limit of *high frequencies* or, equivalently, short bunches.

Diffraction impedance of a round pill-box cavity

Assume axisymmetric geometry, perfectly conducting walls.



Consider a Gaussian beam that is modulated with the wavenumber $k = \omega_0/c$,

$$\tilde{\lambda}(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-z^2/2\sigma_z^2} \cos\left(\frac{\omega_0}{c} z\right)$$

Here $\tilde{\lambda}$ is the modulated fraction of the beam, with $\sigma_z \gg c/\omega_0$.

The field on the wall before the cavity is given by Eq. (2.10)

$$E_\rho = cB_\theta = \frac{1}{4\pi\epsilon_0} \frac{2Q\tilde{\lambda}(z)}{a}$$

The width of the diffraction area is $\sim \sqrt{g\lambda\!\!\!^{-}} = \sqrt{gc/\omega}$ (we use $\lambda\!\!\!^{-} = \lambda/2\pi = 1/k$). Let us estimate the energy loss of the beam ($\tilde{\lambda} \sim 1/\sigma_z$):

$$\Delta\mathcal{E} \sim \frac{\epsilon_0 E^2 + \mu_0 H^2}{2} \times 2\pi a \times \sqrt{g\frac{c}{\omega_0}} \times \sigma_z \sim \epsilon_0 \left(\frac{Q}{\epsilon_0 (2\pi)^{3/2} a \sigma_z}\right)^2 2\pi a \sqrt{g\frac{c}{\omega_0}} \sigma_z$$

Diffraction impedance of pill-box cavity

This energy loss is related to the real part of impedance through Eq. (4.14)

$$\Delta\mathcal{E} = -\frac{Q^2}{\pi}\int_0^\infty d\omega \,\mathrm{Re}\, Z_\ell(\omega)|\hat{\hat{\lambda}}(\omega)|^2$$

We need to calculate $\hat{\hat{\lambda}}(\omega)$

$$\hat{\hat{\lambda}}(\omega) = \frac{1}{2}\left(e^{(\omega-\omega_0)^2\sigma_z^2/2c^2} + e^{(\omega+\omega_0)^2\sigma_z^2/2c^2}\right)$$

Only a narrow peak around $\omega = \omega_0$ contributes to the integral

$$\Delta\mathcal{E} \approx -\frac{cQ^2}{\sqrt{2\pi}\sigma_z}\mathrm{Re}\, Z_\ell(\omega_0)$$

Equating this to the previous expression for $\Delta\mathcal{E}$ we can find the real part of the impedance,

$$\mathrm{Re}\, Z_\ell(\omega) \sim \frac{Z_0}{(2\pi)^{3/2}}\frac{1}{a}\sqrt{\frac{gc}{\omega}}$$

Diffraction impedance of pill-box cavity

More accurate calculations give a numerical factor $\sqrt{2}$ and

$$Z_\ell = \frac{Z_0}{4\pi} \frac{2(1+i)}{\pi^{1/2} a} \sqrt{\frac{gc}{\omega}} \tag{6.19}$$

See movies of field lines in the wake fields.